A CHANGE OF COORDINATES ON THE LARGE PHASE SPACE OF QUANTUM COHOMOLOGY

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ABSTRACT. The Gromov-Witten invariants of a smooth, projective variety V, when twisted by the tautological classes on the moduli space of stable maps, give rise to a family of cohomological field theories and endow the base of the family with coordinates. We prove that the potential functions associated to the tautological ψ classes (the large phase space) and the κ classes are related by a change of coordinates which generalizes a change of basis on the ring of symmetric functions. Our result is a generalization of the work of Manin–Zograf who studied the case where V is a point. We utilize this change of variables to derive the topological recursion relations associated to the κ classes from those associated to the ψ classes.

0. Introduction

Notation . All (co)homology are with $\mathbb{Q}$ coefficients unless explicitly mentioned otherwise. Summation over repeated upper and lower indices is assumed.

The theory of Gromov–Witten invariants of a smooth projective variety V has developed at a rapid pace *c.f.* [4, 6, 32, 45]. These are multilinear operations on the cohomology $H^\bullet(V)$ which can be constructed from intersection numbers on the moduli space of stable maps into V, $\overline{\mathcal{M}}_{g,n}(V)$. In particular, the genus zero Gromov–Witten invariants endow $H^\bullet(V)$ with the structure of the quantum cohomology ring of V. The existence of these invariants was foreseen by physicists who encountered these operations as correlators of a topological sigma model coupled to topological gravity [48]. These invariants are of great

mathematical interest, for example, because they are symplectic invariants of V [36] and because of their close relationship to problems in enumerative geometry [32].

Gromov–Witten invariants satisfy relations (factorization identities) parametrized by the relations between cycles on the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. These relations can be formalized by stating that the space $(H^\bullet(V), \eta)$ (where η is the Poincaré pairing) is endowed with the structure of a cohomological field theory (CohFT) in the sense of Kontsevich–Manin [32]. The Gromov–Witten invariants are characterized by its generating function (the small phase space potential) $\Phi(\mathbf{x})$ where $\mathbf{x} := \{x^\alpha\}$ are coordinates associated to a basis on $H^\bullet(V)$. Restricting to genus zero, $\Phi(\mathbf{x})$ essentially endows $(H^\bullet(V), \eta)$ with the structure of a (formal) Frobenius manifold [11, 22, 38]. It is precisely the structure of a CohFT which was used by Kontsevich–Manin to compute the number of rational curves on $\mathbb{CP}^2$ [32] and the number of elliptic curves by Getzler [17] (where the number is counted with suitable multiplicities).

Furthermore, there are tautological cohomology classes (denoted by ψ_i) associated to the universal curve on $\overline{\mathcal{M}}_{g,n}(V)$ for all $i = 1, \dots, n$ which are the first Chern class of tautological line bundles over $\overline{\mathcal{M}}_{g,n}(V)$. These classes are a generalization of the ψ classes on $\overline{\mathcal{M}}_{g,n}$ due to Mumford. What is remarkable is that by twisting the Gromov–Witten invariants by these ψ classes to obtain the so-called gravitational descendents, one endows $(H^\bullet(V), \eta)$ with the structure of a formal family of CohFT structures whose base is equipped with coordinates $\mathbf{t} := \{t_a^\alpha\}$ where $a \geq 1$ and α is as above. The associated generating function $\mathcal{F}(\mathbf{x}; \mathbf{t})$ (the large phase space potential) reduces to $\Phi(\mathbf{x})$ when $\mathbf{t}$ vanishes. The large phase space potential $\mathcal{F}$ is itself a remarkable object as its exponential is conjectured to satisfy a highest weight condition for the Virasoro algebra [12], a conjecture which has nontrivial consequences [21]. Indeed, when V is a point, this condition is equivalent to the Witten conjecture [48] proven by Kontsevich [31].

There are other tautological cohomology classes on $\overline{\mathcal{M}}_{g,n}(V)$ associated to its universal curve. In this paper, we define generalizations to $\overline{\mathcal{M}}_{g,n}(V)$ of the "modified" κ classes (due to Arbarello–Cornalba [1]) on $\overline{\mathcal{M}}_{g,n}$. We define Gromov–Witten invariants twisted by the κ classes and prove that we obtain a formal family of CohFTs on $(H^\bullet(V), \eta)$ whose base is endowed with coordinates $\mathbf{s} := \{s_a^\alpha\}$ where $a \geq 0$. We then prove that the generating function $\mathcal{G}(\mathbf{x}; \mathbf{s})$ associated to this family can be identified with the large phase space potential $\mathcal{F}(\mathbf{x}; \mathbf{s})$ through an explicit change of variables. This change of variables can

be interpreted as a change of basis in the space of symmetric functions whose variables take values in $H^\bullet(V)$. The variables $\mathbf{s}$ can be interpreted as another canonical set of coordinates on the large phase space. We also utilize this change of variables to derive topological recursion relations for $\mathcal{G}$ in terms of those of $\mathcal{F}$.

When V is convex, the κ classes on $\overline{\mathcal{M}}_{0,n}(V)$ had already been introduced in [24] and the genus zero topological recursion relations were proven. This paper generalizes those results to situations where $\overline{\mathcal{M}}_{g,n}(V)$ need not have the expected dimension (and, hence, the technicalities of the virtual fundamental class cannot be avoided) as well as to derive the change of variables on the large phase space.

When V is a point, our formula reduces to the work of Kaufmann–Manin–Zagier [27] and [40] who noted (see also [34]) that, in addition, the coordinates $\mathbf{s}$ are additive with respect to the tensor product in the category of CohFTs. Manin–Zograf [40] used this formula to compute asymptotic Weil–Peterson volumes of the moduli spaces $\overline{\mathcal{M}}_{g,n}$ as $n \to \infty$ (this was done for $g = 1$ in [23]). However, this additivity property need not hold for a general variety V.

It is worth pointing out several generalizations. First of all, when V is a point, Manin–Zograf use the Witten conjecture to show that their change of variables can be directly interpreted as arising from an analogous change of the cohomology classes appearing in the potential functions. It would be interesting to obtain an analogous result for a general V. Secondly, the above construction should be feasible for any CohFT and there should be coordinates which are additive under tensor product – such a construction would be useful in studying the ring of CohFTs. Work towards this direction is in progress [25]. The third is the fact that there are yet another set of tautological classes (called λ) on $\overline{\mathcal{M}}_{g,n}$ associated to the Hodge bundles. Twisting Gromov–Witten invariants by both the κ and λ classes, one obtains the very large space [23, 40] (see also [13]), a subset of which form coordinates on the moduli space of nondegenerate rank one CohFTs in genus 1 [23] which are additive under tensor product. It would be interesting to understand the role of these additional coordinates for general V.

The first section of the paper is a review of the technicalities necessary to push forward and pull back cohomology classes on the moduli space of stable maps. This includes Gysin morphisms and the flat push-forward.

In the second section, we review the basic properties of the moduli space of stable maps, the structure of the boundary classes, and properties of the virtual fundamental classes.

In the third section we introduce the tautological κ and ψ classes, and prove its restriction properties on the boundary classes.

In the fourth section, we define the notion of a CohFT and its potential function. We review the large phase space potential $\mathcal{F}$. We prove that by introducing the κ classes, $(H^\bullet(V), \eta)$ is endowed with a formal family of CohFT structures together with coordinates on the base of the family.

In the fifth section, we prove that after an explicit change of coordinates, the potential $\mathcal{G}$ can be identified with the large phase space potential $\mathcal{F}$.

In the final section, we derive the topological recursion relations for $\mathcal{G}$ in genus 0 and 1 and derive the usual topological recursion relations for $\mathcal{F}$ through the change of variables.

1. Technical Preliminaries.

In this section we present several technical points needed in the sequel. They are concerned with the Gysin morphisms in homology and cohomology. You may skip this section provided you are willing to accept that everything works at a "naive" level. An article of Fulton and MacPherson [15] may serve as a general reference to this section. All references mentioned in this section deal with schemes rather than stacks, but the sheaf-theoretic approach allows one to work in the category of stacks. If F is a functor, then RF denotes the corresponding derived functor.

Let $\pi : Y \to X$ be a flat representable morphism of Deligne–Mumford stacks with fibers of pure dimension d. As explained in [8] π defines the natural morphism of $Tr_\pi : R^{2d}\mathbb{Q} \to \mathbb{Q}$, which induces the corresponding flat push-forward in cohomology with compact supports $\pi_* : H_c^k(Y) \to H_c^{k-2d}(X)$. (The axioms uniquely defining the morphism Tr are also given in [46] and [35].)

One of the axioms defining the Tr morphism states that it commutes with the base change, that is, with the pull-back on cohomology in a fibered square. However, in this paper we will need to consider commutative squares which are a little more general than the fibered squares. It is the reason for giving the following definition.

Definition 1.1. Let X_1, Y_1, X, Y be Deligne–Mumford stacks. A commutative square

$$
\begin{array}{ccc}
Y_1 & \xrightarrow{f_1} & Y \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\
X_1 & \xrightarrow{f} & X
\end{array}
$$

is called *close to a fibered square* if the induced morphism $g : Y_1 \to X_1 \times_X Y$ is a proper birational morphism, and there is an open subset U of $X_1 \times_X Y$ whose intersection with each fiber of π_1 is a dense subset of the fiber such that $g|_{g^{-1}U}$ is an isomorphism.

Lemma 1.2. *If the commutative square*

$$
\begin{array}{ccc}
Y_1 & \xrightarrow{f_1} & Y \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\
X_1 & \xrightarrow{f} & X
\end{array}
$$

is close to a fibered one, and π and π_1 are representable flat morphisms with fibers of pure dimension d, then $\pi_* f^* = f_1^* \pi_{1*} : H^\bullet(Y) \to H^\bullet(X_1)$.

Proof. Consider the following diagram

$$
\begin{array}{ccccc}
Y_1 & \xrightarrow{g} & X_1 \times_X Y & \xrightarrow{pr_2} & Y \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \pi} \\
X_1 & = = = & X_1 & \xrightarrow{f} & X,
\end{array}
$$

where $pr_2\, g = f_1$. Since the right square is a fibered square it follows from the properties of the Tr morphism that $\pi_* f^* = pr_2^*\, pr_{1*}$. Therefore it remains to show that $pr_{1*} = g^* \pi_{1*}$. It follows from the construction in [8, Sec. 2] that the Tr morphism is determined by a Zariski open subset whose intersection with each fiber is dense. In other words, $Tr : R\, pr_{1!}\, \mathbb{Q} \to \mathbb{Q}$ coincides with

$$
R\, pr_{1!}\, \mathbb{Q} \to R\, pr_{1!}\, Rg_*\, \mathbb{Q} = R\pi_{1!}\, \mathbb{Q} \to \mathbb{Q}.
$$

We have used the fact that $R\, g_* = R\, g_!$ since g is proper. $\qquad \square$

Dually, a flat morphism $\pi : Y \to X$ with fibers of pure dimension d determines a flat pull-back $\pi^* : H_k(X) \to H_{k+2d}(Y)$. It is shown in [35, Sec. 6] that π^* agrees with the flat pull-back $\pi^* : A_k(X) \to A_{k+d}(Y)$ via the cycle map. (In the set up of bivariant intersection theory [15, 2.3] each flat morphism π determines a canonical element in $T^{-2d}(Y \to X)$.)

We also need to define the Gysin morphisms associated to regular imbeddings. A closed imbedding $i : X_1 \to X$ is called a *regular imbedding of codimension* d if the conormal sheaf of X_1 in X is a locally free sheaf on X_1 of rank d [14, B.7.1]. Let $i : X_1 \to X$ be a regular embedding of codimension d. The corresponding canonical element $\theta_i \in H^{2d}(X, X - X_1)$ is constructed in [3, IV.4] and [47, Sec. 5]. (In bivariant intersection theory $H^{2d}(X, X - X_1) = T^{2d}(X_1 \to X)$.) If

$$
\begin{array}{ccc}
Y_1 & \xrightarrow{\ i_1\ } & Y \\
\scriptstyle f_1 \downarrow & & \scriptstyle f \downarrow \\
X_1 & \xrightarrow{\ i\ } & X
\end{array}
$$

is a fibered square, then the pull-back $f^*\theta_i$ determines an element in $H^{2d}(Y, Y - Y_1)$. Accordingly, it defines Gysin homomorphisms:

$$i^! : H_k(Y) \to H_{k-2d}(Y_1) \quad \text{and}$$
$$i_! : H^k(Y_1) \to H^{k+2d}(Y)$$

by the cap-product (or cup-product) with $f^*\theta_i$. However, we will denote $i^! : H_k(X) \to H_{k-2d}(X_1)$ by i^*, and $i_! : H^k(X_1) \to H^{k+2d}(X)$ by i_*. This agrees with the notation from [14]. If $E \to X$ is a rank d vector bundle, and X_1 is the zero scheme of a section $i : X \to E$, then $i_*1 = c_d E$ [14, Sec. 19.2]. The Gysin morphism $i^!$ defined above agrees with the Gysin morphism $i^!$ on the level of Chow groups via the cycle map [47].

Remark . More generally, one can define the Gysin morphisms for local complete intersection morphisms. If a flat morphisms is at the same time a local complete interesection morphism, then two definitions agree.

The morphisms $\pi_*, \pi^*, i_!, i^!$ satisfy the expected projection type formulae and commute with the standard pull-backs and push-forwards [15, 2.5]. We will use these properties without explicitly mentioning them.

2. The Moduli Spaces of Stable Maps

We adopt the notation from [18]. Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable curves. The stability implies that $2g - 2 + n > 0$. Let Γ be a stable graph of genus g with n tails. We denote by $\overline{\mathcal{M}}(\Gamma) \subset \overline{\mathcal{M}}_{g,n}$ the

closure in $\overline{\mathcal{M}}_{g,n}$ of the locus of stable curves with the dual graph Γ, and by i_Γ the corresponding inclusion. Let

$$\widetilde{\mathcal{M}}(\Gamma) := \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}.$$

Then $\mathrm{Aut}(\Gamma)$ acts on $\widetilde{\mathcal{M}}(\Gamma)$. The natural morphism

$$\mu_\Gamma : \widetilde{\mathcal{M}}(\Gamma) \to \overline{\mathcal{M}}(\Gamma)$$

identifies $\overline{\mathcal{M}}(\Gamma)$ with $\widetilde{\mathcal{M}}(\Gamma)/\mathrm{Aut}(\Gamma)$. We denote by ρ_Γ the composition

$$\rho_\Gamma : \widetilde{\mathcal{M}}(\Gamma) \xrightarrow{\mu_\Gamma} \overline{\mathcal{M}}(\Gamma) \xrightarrow{i_\Gamma} \overline{\mathcal{M}}_{g,n}.$$

The previous considerations apply word for word to the moduli spaces of prestable curves $\overline{\mathsf{M}}_{g,n}$, $g \geq 0, n \geq 0$, their subspaces $\overline{\mathsf{M}}(\Gamma)$, and the products $\widetilde{\mathsf{M}}(\Gamma)$ [18, Sec. 2]. Note that $\overline{\mathcal{M}}_{g,n}$ is an open dense substack of $\overline{\mathsf{M}}_{g,n}$ when $2g - 2 + n > 0$, and, more generally, $\overline{\mathcal{M}}(\Gamma)$ is an open dense substack of $\overline{\mathsf{M}}(\Gamma)$ when Γ is a stable graph.

We adopt a similar notation for the substacks of $\overline{\mathcal{M}}_{g,n}(V, \beta)$ determined by decorated stable graphs. Let $H_2^+(V, \mathbb{Z})$ denote the semigroup generated by those homology classes represented by the image of a morphism from a curve into V. Let $\mathbb{G}$ be a stable graph of genus g with n tails whose vertices are decorated by elements of $H_2^+(V, \mathbb{Z})$. (Henceforth, such decorated graphs will be denoted by $\mathbb{G}$.) Then we denote by $\overline{\mathcal{M}}(\mathbb{G}, V)$ the closure in $\overline{\mathcal{M}}_{g,n}(V, \beta)$ of those points in the moduli space of stable maps whose dual graph is $\mathbb{G}$. Let $\widetilde{\mathcal{M}}(\mathbb{G}, V)$ be determined by the following fibered square (cf. [18, Sec. 6]):

$$
\begin{array}{ccc}
\widetilde{\mathcal{M}}(\mathbb{G}, V) & \xrightarrow{\Delta} & \displaystyle\prod_{v \in V(\mathbb{G})} \overline{\mathcal{M}}_{g(v),n(v)}(V, \beta(v)) \\
\downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} \\
V^{E(\mathbb{G})} & \xrightarrow{\Delta_1} & V^{E(\mathbb{G})} \times V^{E(\mathbb{G})},
\end{array}
$$

where Δ_1 is the diagonal morphism. In the sequence

$$\prod_{v \in V(\mathbb{G})} \overline{\mathcal{M}}_{g(v),n(v)}(V, \beta(v)) \xleftarrow{\Delta} \widetilde{\mathcal{M}}(\mathbb{G}, V) \xrightarrow{\mu(\mathbb{G})} \overline{\mathcal{M}}(\mathbb{G}, V) \xrightarrow{i(\mathbb{G})} \overline{\mathcal{M}}_{g,n}(V, \beta),$$

the morphism $\mu(\mathbb{G})$ is the quotient by $\mathrm{Aut}(\mathbb{G})$ identifying $\overline{\mathcal{M}}(\mathbb{G}, V)$ with the quotient, and $i(\mathbb{G})$ is the inclusion of a substack. We denote the composition of two morphisms on the right by $\rho(\mathbb{G})$.

We will also need to introduce some other notation to describe the pull back of the virtual fundamental classes with respect to the inclusions of the strata (cf. [18, Sec. 6]. Let Γ be a graph of genus g with n tails, not necessarily stable. We define

$$\overline{\mathcal{M}}(\Gamma, V, \beta) := \overline{\mathcal{M}}_{g,n}(V, \beta) \times_{\overline{\mathsf{M}}_{g,n}} \overline{\mathsf{M}}(\Gamma).$$

It is the closure of the subset of $\overline{\mathcal{M}}_{g,n}(V, \beta)$ whose points correspond to the graph Γ after forgetting the decoration. If Γ is a stable graph, then $\overline{\mathcal{M}}(\Gamma, V, \beta) = \overline{\mathcal{M}}_{g,n}(V, \beta) \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}(\Gamma)$ since $\overline{\mathcal{M}}(\Gamma)$ is dense in $\overline{\mathsf{M}}(\Gamma)$. If $\mathbb{G}$ is a decorated graph we denote by $\mathbb{G}^0$ the underlying non-decorated graph. Let

$$\widetilde{\mathcal{M}}(\Gamma, V, \beta) := \coprod_{\mathbb{G}:\mathbb{G}^0=\Gamma} \widetilde{\mathcal{M}}(\mathbb{G}, V).$$

As before, $\Delta : \widetilde{\mathcal{M}}(\Gamma, V, \beta) \to \coprod_{\mathbb{G}^0=\Gamma} \prod_{v \in V(\mathbb{G})} \overline{\mathcal{M}}_{g(v),n(v)}(V, \beta(v))$ is determined by the diagonal morphism. One has the natural morphism:

$$\rho(\Gamma) : \widetilde{\mathcal{M}}(\Gamma, V, \beta) \xrightarrow{\mu(\Gamma)} \overline{\mathcal{M}}(\Gamma, V, \beta) \xrightarrow{i(\Gamma)} \overline{\mathcal{M}}_{g,n}(V, \beta).$$

Here $i(\Gamma)$ is an inclusion of a substack, and $\mu(\Gamma)$ factors as

$$\widetilde{\mathcal{M}}(\Gamma, V, \beta) \to \widetilde{\mathcal{M}}(\Gamma, V, \beta)/\operatorname{Aut}(\Gamma) \to \overline{\mathcal{M}}(\Gamma, V, \beta),$$

where the second morphism is a proper, surjective, birational morphism. The difference with the previous situation is explained by the fact that two substacks $\overline{\mathcal{M}}(\mathbb{G}, \beta)$ and $\overline{\mathcal{M}}(\mathbb{G}', \beta)$ of $\overline{\mathcal{M}}_{g,n}(V, \beta)$ whose underlying undecorated graphs are the same may have a nonempty intersection.

3. Tautological Classes

In this section we introduce the tautological κ classes on the moduli spaces of stable maps which generalize the corresponding tautological classes on the moduli spaces of stable curves. We will also show how these classes restrict to the boundary strata.

Let $\pi : \overline{\mathcal{M}}_{g,n+1}(V, \beta) \to \overline{\mathcal{M}}_{g,n}(V, \beta)$ be the universal curve. We assume that π "forgets" the $(n+1)^{\text{st}}$ marked point. The morphism π has n canonical sections $\sigma_1, \ldots, \sigma_n$. Each of these sections determines a regular embedding. We denote by ω the relative dualizing sheaf of π.

Definition 3.1. For each $i = 1, \ldots, n$ the *tautological line bundle* $\mathcal{L}_i$ on $\overline{\mathcal{M}}_{g,n}(V, \beta)$ is $\sigma_i^*\omega$. The *tautological class* $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}(V, \beta))$ is the first Chern class $c_1(\mathcal{L}_i)$.

Remark . It is shown in [18, Sec. 5] that $\psi_i = p^*\Psi_i$, where $p : \overline{\mathcal{M}}_{g,n}(V, \beta) \to \overline{\mathsf{M}}_{g,n}$, where Ψ_i is the tautological class in $H^2(\overline{\mathsf{M}}_{g,n})$.

One can also pull back cohomology classes from V to $\overline{\mathcal{M}}_{g,n}(V,\beta)$ using the evaluation maps to obtain the Gromov–Witten classes. The definition of the κ classes involves both, powers of the ψ classes and these pull backs.

Definition 3.2. The *tautological class κ_a* in $H^\bullet(\overline{\mathcal{M}}_{g,n}(V,\beta)) \otimes H^\bullet(V)^*$ for $a \geq -1$ is defined as follows. For each $\gamma \in H^\bullet(V)$, the cohomology class $\kappa_a(\gamma)$ is $\pi_*(\psi_{n+1}^{a+1} ev_{n+1}^*(\gamma))$, where π is the universal curve defined above. In particular, if γ has definite degree $|\gamma|$ then $\kappa_a(\gamma)$ has degree $2a + |\gamma|$. If $\{e_\alpha\}_{\alpha \in A}$, is a homogeneous basis for $H^\bullet(V)$, then $\kappa_{a,\alpha}$ denotes the cohomology class $\kappa_a(e_\alpha)$.

Remark . The class $\kappa_{-1}(\gamma)$ vanishes due to dimensional reasons if $|\gamma| < 2$. In addition, all classes $\kappa_{-1}(\gamma)$ vanish on $\overline{\mathcal{M}}_{g,n}(V,0)$. The classes $\kappa_{-1}(\gamma)$ are not needed in the change of coordinates formula in Sec. 5.

Our definition corresponds to the "modified" κ classes defined by Arbarello and Cornalba [1] rather than the "classical" κ classes defined by Mumford [41].

The following lemma shows how the κ classes restrict to the boundary substacks of $\overline{\mathcal{M}}_{g,n}(V,\beta)$.

Lemma 3.3. *Let $\mathbb{G}$ be a stable $H_2^+(V,\mathbb{Z})$ decorated genus g, degree β graph with n tails. Denote the class $\kappa_a(\gamma)$ on $\overline{\mathcal{M}}_{g,n}(V,\beta)$ (resp. $\overline{\mathcal{M}}(v)$, where $v \in V(\mathbb{G})$) by κ (resp. κ_v). Then*

$$\rho(\mathbb{G})^*(\kappa) = \Delta^* \sum_{v \in V(\mathbb{G})} \kappa_v.$$

Proof. Let $v \in V(\mathbb{G})$. Denote by $\mathbb{G}(v)$ the graph obtained from $\mathbb{G}$ by attaching a tail labeled $n+1$ to the vertex v of $\mathbb{G}$. For each $v \in V(\mathbb{G})$ the graph $\mathbb{G}(v)$ determines a substack of $\overline{\mathcal{M}}_{g,n+1}(V,\beta)$, and there are natural morphisms

$$\widetilde{\pi} : \prod_{w \in V(\mathbb{G}(v))} \overline{\mathcal{M}}(w) \to \prod_{w \in V(\mathbb{G})} \overline{\mathcal{M}}(w), \text{ and } \widetilde{\pi} : \widetilde{\mathcal{M}}(\mathbb{G}(v),V) \to \widetilde{\mathcal{M}}(\mathbb{G},V).$$

Consider the following commutative diagram

$$\coprod_{v \in V(\mathbb{G})} \prod_{w \in \mathbb{G}(v)} \overline{\mathcal{M}}(w) \xleftarrow{\coprod \Delta} \coprod_{v \in V(\mathbb{G})} \widetilde{\mathcal{M}}(\mathbb{G}(v),V) \xrightarrow{\coprod \rho(\mathbb{G}')} \overline{\mathcal{M}}_{g,n+1}(V,\beta)$$

$$\downarrow {\coprod \widetilde{\pi}} \qquad\qquad \downarrow {\coprod \widetilde{\pi}} \qquad\qquad \downarrow \pi$$

$$\prod_{w \in \mathbb{G}} \overline{\mathcal{M}}(w) \xleftarrow{\quad\Delta\quad} \widetilde{\mathcal{M}}(\mathbb{G},V) \xrightarrow{\rho(\mathbb{G})} \overline{\mathcal{M}}_{g,n}(V,\beta).$$

Note that the left square is a fibered square, the right square is close to a fibered square in the sense of Def. 1.1, and all morphisms $\pi, \widetilde{\pi}$ are

representable and flat. Therefore, one can apply Lem. 1.2. Also note that for each $v \in V(\mathbb{G})$ one has

$$\rho(\mathbb{G}(v))^* \psi_{n+1} = \Delta^* \psi_{n+1} \quad \text{and} \quad \rho(\mathbb{G}(v))^* ev_{n+1}^* \gamma = \Delta^* ev_{n+1}^* \gamma.$$

Now

$$\rho(\mathbb{G})^*(\kappa) = \rho(\mathbb{G})^* \pi_*(\psi_{n+1}^{a+1} ev_{n+1}^* \gamma) = \sum_{v \in V(\mathbb{G})} \widetilde{\pi}_* \rho(\mathbb{G}(v))^*(\psi_{n+1}^{a+1} ev_{n+1}^* \gamma)$$

$$= \sum_{v \in V(\mathbb{G})} \widetilde{\pi}_* \Delta^*(\psi_{n+1}^{a+1} ev_{n+1}^* \gamma) = \sum_{v \in V(\mathbb{G})} \Delta^* \widetilde{\pi}_*(\psi_{n+1}^{a+1} ev_{n+1}^* \gamma) = \Delta^* \sum_{v \in V(\mathbb{G})} \kappa_v.$$

$\square$

The above lemma shows that the class $\kappa_a(\gamma)$ restricts to the sum of the $\kappa_a(\gamma)$ classes. It follows that $\exp(\kappa_a(\gamma))$ restricts to the product of $\exp(\kappa_a(\gamma))$. More generally, $\exp(\sum_{a=-1}^{\infty} \kappa_{a,\alpha} s_a^\alpha)$, where s_i's are formal variables, restricts to the product of $\exp(\sum_{a=-1}^{\infty} \kappa_{a,\alpha} s_a^\alpha)$. This will be used in Sec. 4.

4. Cohomological Field Theories

In this section, we define a cohomological field theory in the sense of Kontsevich–Manin [32]. We prove that the Gromov–Witten invariants twisted by the κ classes endows $H^\bullet(V)$ together with its Poincaré pairing with the a family of CohFT structures. In genus zero, this reduces to endowing $H^\bullet(V)$ with a family of formal Frobenius manifold structures arising from the Poincaré pairing and deformations of the cup product on $H^\bullet(V)$. These deformations contain quantum cohomology as a special case.

Definition 4.1. Let $(\mathcal{H}, \eta)$ be an r-dimensional vector space $\mathcal{H}$ with an even, symmetric nondegenerate, bilinear form η. A *(complete) rank r cohomological field theory (or CohFT) with state space $(\mathcal{H}, \eta)$* is a collection $\Omega := \{\Omega_{g,n}\}$ where $\Omega_{g,n}$ is an even element in $\mathcal{R}_{g,n} := H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes T^n \mathcal{H}^*$ defined for stable pairs (g, n) satisfying (i) to (iii) below (where the summation convention has been used):

i: $\Omega_{g,n}$ is invariant under the diagonal action of the symmetric group S_n on $T^n \mathcal{H}$ and $\overline{\mathcal{M}}_{g,n}$.

ii: For each partition of $[n] = J_1 \sqcup J_2$ such that $|J_1| = n_1$ and $|J_2| = n_2$ and nonnegative g_1, g_2 such that $g = g_1 + g_2$ and $2g_i - 2 + n_i + 1 > 0$ for all i, consider the inclusion map $\rho : \overline{\mathcal{M}}_{g_1, J_1 \sqcup *} \times \overline{\mathcal{M}}_{g_2, J_2 \sqcup *} \to \overline{\mathcal{M}}_{g_1 + g_2, n}$ where $*$ denotes the two marked

points that are attached under the inclusion map. The forms satisfy the restriction property

$$\rho^* \, \Omega_{g,n}(\gamma_1, \gamma_2 \ldots, \gamma_n) =$$
$$\pm \, \Omega_{g_1,n_1}((\bigotimes_{\alpha \in J_1} \gamma_\alpha) \otimes e_\mu) \, \eta^{\mu\nu} \otimes \Omega_{g_2,n_2}(e_\nu \otimes \bigotimes_{\alpha \in J_2} \gamma_\alpha)$$

where the sign $\pm$ is the usual one obtained by applying the permutation induced by the partition to $(\gamma_1, \gamma_2, \ldots \gamma_n)$ taking into account the grading of $\{\gamma_i\}$ and where $\{e_\alpha\}$ is a homogeneous basis for $\mathcal{H}$.

iii: Let $\rho_0 : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ be the canonical map corresponding to attaching the last two marked points together then

$$\rho_0^* \, \Omega_{g,n}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \Omega_{g-1,n+2}(\gamma_1, \gamma_2, \ldots, \gamma_n, e_\mu, e_\nu) \, \eta^{\mu\nu}.$$

iv: If, in addition, there exists an even element e_0 in $\mathcal{H}$ such that

$$\pi^* \, \Omega_{g,n}(\gamma_1, \ldots, \gamma_n) = \Omega_{g,n+1}(\gamma_1, \ldots, \gamma_n, e_0)$$

and

$$\fint_{\overline{\mathcal{M}}_{0,3}} \Omega_{0,3}(e_0, \gamma_1, \gamma_2) = \eta(\gamma_1, \gamma_2)$$

for all γ_i in $\mathcal{H}$ then Ω endows $(\mathcal{H}, \eta)$ with the structure of a *CohFT with flat identity* e_0.

A *cohomological field theory of genus g* consists of only those $\Omega_{g',n}$ where $g' \leq g$ which satisfy the subset of axioms of a cohomological field theory which includes only objects of genus $g' \leq g$.

The strata maps ρ and ρ_0 in the above definition can be extended to arbitrary boundary strata on $\overline{\mathcal{M}}_{g,n}$. Let Γ be a stable graph then there is a canonical map ρ_Γ obtained by composition of the canonical maps

$$\prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}.$$

Since the map ρ_Γ can be constructed from morphisms in (ii) and (iii) above, $\Omega_{g,n}$ satisfies a restriction property of the form

$$(1) \qquad \rho_\Gamma^* \Omega_{g,n} = \eta_\Gamma^{-1}(\bigotimes_{v \in V(\Gamma)} \Omega_{g(v),n(v)})$$

where

$$\eta_\Gamma^{-1} : \bigotimes_{v \in V(\Gamma)} \mathcal{R}_{g(v),n(v)} \to \mathcal{R}_{g,n}$$

is the linear map contracting tensor factors of $\mathcal{H}$ using the metric η induced from successive application of equations (ii) and (iii) above.

Notice that the definition of a cohomological field theory is valid even when enlarges the ground ring from $\mathbb{C}$ to another ring $\mathcal{K}$.

Finally, axioms (i) to (iii) in the definition of a CohFT is equivalent to endowing $(\mathcal{H}, \eta)$ with the structure of an algebra over the modular operad $H_\bullet(\overline{\mathcal{M}}) := \{ H_\bullet(\overline{\mathcal{M}}_{g,n}) \}$ [20].

Definition 4.2. Let Λ consist of formal symbols q^β for all $\beta \in H_2^+(V, \mathbb{Z})$ together with the multiplication $(q^\beta q^{\beta'}) \mapsto q^{\beta + \beta'}$. Let $\mathbb{C}[[\Lambda]]$ consist of formal sums $\sum_{\beta \in H_2^+(V, \mathbb{Z})} a_\beta q^\beta$ where a_β are elements in $\mathbb{C}$. Assign to each q^β, the degree $-2c_1(V) \cap \beta$. The product is well-defined according to [30, Prop. II.4.8]. This endows Λ with the structure of a semigroup with unit. Furthermore, let $\mathbb{C}[[\Lambda, \mathbf{s}]] := \mathbb{C}[[\Lambda]][[\mathbf{s}]]$, formal power series in the variables $\mathbf{s}$ with coefficients in $\mathbb{C}[[\Lambda]]$.

Notation . Let V be a topological space and let $H^\bullet(V, \mathbb{C})$ be given a homogeneous basis $\mathbf{e} := \{ e_\alpha \}_{\alpha \in A}$ and let e_0 denote the identity element. Let $\mathbf{s} := \{ s_a^\alpha \,|\, a \geq -1, \alpha \in A \}$ be a collection of formal variables with grading $|s_a^\alpha| = 2a + |e_\alpha|$. All formal power series and polynomials in a collection of variables (*e.g.* $\mathbf{s}$) are in the $\mathbb{Z}_2$-graded sense.

It will be useful to associate a generating function (called the *potential*) to each CohFT.

Definition 4.3. Let Ω be a rank r CohFT with state space $(\mathcal{H}, \eta)$. Its *potential function* Φ in $\lambda^{-2} \mathbb{C}[[\mathcal{H}, \lambda]]$ is defined by

$$\Phi(\mathbf{x}) := \sum_{g \geq 0} \Phi_g(\mathbf{x}) \, \lambda^{2g-2}$$

where

$$\Phi(\mathbf{x}) := \sum_{n=3}^{\infty} \frac{1}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})$$

and $\mathbf{x} = \sum_{\alpha=0}^{r-1} x^\alpha e_\alpha$ for a given homogeneous basis $\{ e_0, \dots, e_{r-1} \}$ for $\mathcal{H}$. The formal parameter λ is even.

In genus zero, the potential function yields yet another formulation of a CohFT which is essentially the definition of a formal Frobenius manifold structure on its state space.

Theorem 4.4. *Let $(\mathcal{H}, \eta)$ be an r dimensional vector space with metric. An element $\Phi_0(\mathbf{x})$ in $\mathbb{C}[[\mathcal{H}]]$ is the potential of a rank r, genus zero CohFT $(\mathcal{H}, \eta)$ if and only if* [32, 38] *it contains only terms which are*

cubic and higher order in the variables $x^0, \ldots, x^r$ *and it satisfies the WDVV equation*

$$(\partial_a \partial_b \partial_e \Phi_0) \, \eta^{ef} \, (\partial_f \partial_c \partial_d \Phi_0) = (-1)^{|x_a|(|x_b|+|x_c|)} \, (\partial_b \partial_c \partial_e \Phi_0) \, \eta^{ef} \, (\partial_f \partial_a \partial_d \Phi_0),$$

where $\eta_{ab} := \eta(e_a, e_b)$, η^{ab} *is in inverse matrix to* η_{ab}, ∂_a *is derivative with respect to* x^a, *and the summation convention has been used. Furthermore, any genus zero CohFT is completely characterized by its genus zero potential* $\Phi_0(\mathbf{x})$.

The theorem follows from the work of Keel [28] who proved that all relations between boundary divisors on $\overline{\mathcal{M}}_{0,n}$ arise from lifting the basic codimension one relation on $\overline{\mathcal{M}}_{0,4}$.

As before, one can extend the ground ring $\mathbb{C}$ above to $\mathbb{C}[[\Lambda, \mathbf{s}]]$ in the definition of the potential of a genus zero CohFT and the above theorem extends, as well. In our setting, the potential is a formal function on $\mathcal{H} := H^\bullet(V, \mathbb{C}[[\Lambda, \mathbf{s}]])$ and η is the Poincaré pairing extended linearly to $\mathbb{C}[[\Lambda, \mathbf{s}]]$. Φ belongs to $\lambda^{-2} \mathbb{C}[[\Lambda, \mathbf{s}, \lambda]][[x^0, \ldots, x^r]]$ Again, if $H^\bullet(V)$ consists entirely of even dimensional classes then plugging in numbers (almost all of which are zero) for all s_a^α where $a = -1, 0, 1, \ldots$ and $\alpha = 0, 1, \ldots, r-1$ and setting $\lambda = 1$, one obtains families of CohFT structures on $H^\bullet(V, \mathbb{C}[[\Lambda]])$.

Notation . We define $\mathbf{s}\boldsymbol{\kappa}$ to be $\displaystyle\sum_{a=-1}^{\infty} \kappa_{a,\alpha} s_a^\alpha$. Note that each term has even parity.

Theorem 4.5. *Let* V *be a smooth projective variety. For each pair* (g, n) *such that* $2g - 2 + n > 0$, *let* $\Omega_{g,n}$ *be the element of* $\mathcal{R}_{g,n}(V)[[\Lambda, \mathbf{s}]]$ *defined by*

$$\Omega_{g,n}(\gamma_1, \ldots, \gamma_n)$$
$$:= \sum_{\beta \in H_2^+(V, \mathbb{Z})} \mathrm{st}_*(\, ev_1^* \gamma_1 \cdots ev_n^* \gamma_n \, \exp(\boldsymbol{\kappa} \mathbf{s}) \cap [\overline{\mathcal{M}}_{g,n}(V, \beta)]^{virt}) \, q^\beta,$$

where $\gamma_1, \gamma_2, \ldots, \gamma_n$ *are elements in* $H^\bullet(V, \mathbb{C})$. *Then* $\Omega := \{ \Omega_{g,n} \}$ *endows* $(H^\bullet(V, \mathbb{C}[[\Lambda, \mathbf{s}]]), \eta)$ *with the structure of a CohFT where* η *is the Poincaré pairing extended* $\mathbb{C}[[\Lambda, \mathbf{s}]]$-*linearly.*

Proof. It is clear that the morphisms $\Omega_{g,n}$ are S_n-equivariant. In order to prove the restriction properties fix $\beta \in H_2^+(V, \mathbb{Z})$, (g, n) such that $2g - 2 + n > 0$, and a stable graph Γ of genus g with n tails. Let G be the set of all $H_2^+(V, \mathbb{Z})$ decorated graphs such that the underlying

graph without decoration is Γ. Let

$$X := \coprod_{\mathbb{G} \in G} X_{\mathbb{G}}, \quad \text{where} \quad X_{\mathbb{G}} := \prod_{v \in \mathbb{G}} \overline{\mathcal{M}}(v),$$

and let $[X_{\mathbb{G}}]^{virt} \in H_{\bullet}(X_{\mathbb{G}})$ be the product of the corresponding virtual fundamental classes. Consider the following commutative diagram:

$$
\begin{array}{ccccccc}
X & \xleftarrow{\ \Delta\ } & \widetilde{\mathcal{M}}(\Gamma, V, \beta) & \xrightarrow{\ \mu(\Gamma)\ } & \overline{\mathcal{M}}(\Gamma, V, \beta) & \xrightarrow{\ i(\Gamma)\ } & \overline{\mathcal{M}}_{g,n}(V, \beta) \\
\downarrow{\scriptstyle \text{st}} & & \downarrow{\scriptstyle \text{st}} & & \downarrow{\scriptstyle \text{st}} & & \downarrow{\scriptstyle \text{st}} \\
\widetilde{\mathcal{M}}(\Gamma) & =\!=\!= & \widetilde{\mathcal{M}}(\Gamma) & \xrightarrow{\ \mu_{\Gamma}\ } & \overline{\mathcal{M}}(\Gamma) & \xrightarrow{\ i_{\Gamma}\ } & \overline{\mathcal{M}}_{g,n}.
\end{array}
$$

We want to see how the β summand of $\Omega_{g,n}$ restricts to $H^{\bullet}(\widetilde{\mathcal{M}}(\Gamma))$.

In the sequence of equations below we will use the following properties. The right square of the above diagram is a fibered square. All vertical morphisms st are proper. If $x \in H^{\bullet}(\widetilde{\mathcal{M}}(\Gamma))$ is invariant under the action of $\mathrm{Aut}(\Gamma)$, then $\mu_{\Gamma}^* \mu_{\Gamma *} x = N x$, where $N := |\mathrm{Aut}(\Gamma)|$. In addition, we use the following result of Getzler [18, Thm. 13]:

$$i_{\Gamma}^! [\overline{\mathcal{M}}_{g,n}(V, \beta)]^{virt} = \frac{1}{N} \mu(\Gamma)_* \Delta_1^! \sum_{\mathbb{G} \in G} [X_{\mathbb{G}}]^{virt},$$

where

$$\Delta_1 : V^{E(\mathbb{G})} \to V^{E(\mathbb{G})} \times V^{E(\mathbb{G})}$$

is the diagonal morphism.

If $\mathbb{G} \in G$, and $v \in V(\mathbb{G})$, then we denote by $\boldsymbol{\gamma}_v$ the tensor product of the corresponding γ_i's on $\overline{\mathcal{M}}(v)$, and by $\boldsymbol{\kappa}_v \mathbf{s}$ the formal sum on $\overline{\mathcal{M}}(v)$. For the sake of brevity we will write μ for μ_{Γ}, i for i_{Γ}, and $\boldsymbol{\gamma}$ for $\otimes \gamma_i$. The sums below are always taken over $\mathbb{G} \in G$.

$$\mu^* i^* st_* (ev^* \boldsymbol{\gamma} \exp(\boldsymbol{\kappa}\mathbf{s}) \cap [\overline{\mathcal{M}}_{g,n}(V, \beta)]^{virt})$$

$$= \mu^* \, \mathrm{st}_* \, i^! (ev^* \boldsymbol{\gamma} \exp(\boldsymbol{\kappa}\mathbf{s}) \cap [\overline{\mathcal{M}}_{g,n}(V, \beta)]^{virt})$$

$$= \mu^* \, \mathrm{st}_* (i(\Gamma)^* (ev^* \boldsymbol{\gamma} \exp(\boldsymbol{\kappa}\mathbf{s})) \cap i^! [\overline{\mathcal{M}}_{g,n}(V, \beta)]^{virt})$$

$$= \mu^* \, \mathrm{st}_* (i(\Gamma)^* (ev^* \boldsymbol{\gamma} \exp(\boldsymbol{\kappa}\mathbf{s})) \cap \frac{1}{N} \mu(\Gamma)_* \Delta_1^! \sum [X_{\mathbb{G}}]^{virt})$$

$$= \frac{1}{N} \mu^* \, \mathrm{st}_* \, \mu(\Gamma)_* (\mu(\Gamma)^* i(\Gamma)^* (ev^* \boldsymbol{\gamma} \exp(\boldsymbol{\kappa}\mathbf{s})) \cap \Delta_1^! \sum [X_{\mathbb{G}}]^{virt})$$

$$= \mathrm{st}_* \sum (\Delta^* (\otimes_{v \in \mathbb{G}} ev^* \boldsymbol{\gamma}_v \exp(\boldsymbol{\kappa}_v \mathbf{s})) \cap \Delta_1^! \sum [X_{\mathbb{G}}]^{virt})$$

$$= \sum \otimes_{v \in \mathbb{G}} \mathrm{st}_* (ev^* \boldsymbol{\gamma}_v \exp(\boldsymbol{\kappa}_v \mathbf{s}) \cap \Delta_* \Delta_1^! [X_{\mathbb{G}}]^{virt}).$$

Summing over all β gives the statement of the theorem taking into account that $\Delta_* \Delta_1^!$ is the cap-product with the Poincaré dual of the diagonal in $V^{E(\Gamma)} \times V^{E(\Gamma)}$. $\qquad\qquad\qquad\qquad\square$

Th. 4.5 provides a CohFT determined by the κ classes. One can similarly construct a CohFT determined by the ψ classes. Its potential is the usual potential. A more general construction will appear in [25].

Remark . The potential of the CohFT defined in the previous theorem coincides with the usual notion of potential of Gromov–Witten invariants up to terms quadratic in the variables $\mathbf{x}$ which correspond to contributions from the moduli spaces $\overline{\mathcal{M}}_{g,n}(V)$ where $2g-2+n \leq 0$.

5. The Change of Coordinates

In this section we prove the change of coordinate formula on the large phase space. Throughout the rest of this section, we fix a homogeneous basis $\{e_\alpha\}$ where $\alpha \in A$ of $H^\bullet(V)$ such that e_0 is the identity element. We also fix a total ordering on A.

Remark . In this section we will not use the tautological classes $\kappa_{-1}(\gamma)$.

Definition 5.1. Let $\beta \in H_2^+(V, \mathbb{Z})$, and e_α, $\alpha = 0, \ldots, r-1$ be a basis of $H^\bullet(V)$. Assume that all $d_i > 0$ and all $a_i \geq 0$. We define

$$\langle \sigma_{\nu_1} \ldots \sigma_{\nu_n}\, \tau_{d_1,\mu_1} \ldots \tau_{d_k,\mu_k}\, \kappa_{a_1,\alpha_1} \ldots \kappa_{a_l,\alpha_l} \rangle_{g,\beta} :=$$

$$\int_{[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{virt}} ev_1^*(e_{\nu_1}) \ldots ev_n^*(e_{\nu_n})$$

$$\pi_*(\psi_{n+1}^{d_1} ev_{n+1}^*(e_{\mu_1}) \ldots \psi_{n+k}^{d_k} ev_{n+k}^*(e_{\mu_k})) \kappa_{a_1,\alpha_1} \ldots \kappa_{a_l,\alpha_l},$$

where $\pi : \overline{\mathcal{M}}_{g,n+k}(V,\beta) \to \overline{\mathcal{M}}_{g,n}(V,\beta)$ "forgets" the last k marked points.

Remark . This definition differs from the standard one. However, if no κ classes are present, then the intersection number above is the standard intersection number of the ψ and the pull-back classes with $[\overline{\mathcal{M}}_{g,n+k}(V,\beta)]^{virt}$. This definition is motivated by the representation of the large phase space on the level of cohomology classes in Sec. 4. Also, it will be easier to work with this definition to derive the coordinate change below.

Let the sequence $\nu_1, \ldots, \nu_n$ contain r_ν elements ν, $\nu \in A$, the sequence $(d_1, \mu_1), (d_2, \mu_2), \ldots, (d_k, \mu_k)$ contain $m_{d,\mu}$ pairs (d, μ), where $d > 0$, $\mu \in A$, and the sequence $(a_1, \alpha_1), (a_2, \alpha_2), \ldots, (a_l, \alpha_l)$ contain $p_{a,\alpha}$ pairs (a, α) where $a \geq 0$, $\alpha \in A$. Then we also denote the intersection number above by $\langle \boldsymbol{\sigma}^{\mathbf{r}} \boldsymbol{\tau}^{\mathbf{m}} \boldsymbol{\kappa}^{\mathbf{P}} \rangle_{g,\beta}$. One has to be careful if $H^\bullet(V)$

has elements of odd degree. In this case $\langle \boldsymbol{\sigma^r \tau^m \kappa^P} \rangle_{g,\beta}$ denotes the intersection number above with the following ordering. If $i < j$ then, using the chosen order on A,

a) $\nu_i \leq \nu_j$;
b) $d_i < d_j$, or $d_i = d_j$ and $\mu_i \leq \mu_j$;
c) $a_i < a_j$, or $a_i = a_j$ and $\alpha_i \leq \alpha_j$.

Definition 5.2. We define

$$\langle \boldsymbol{\sigma^r \tau^m \kappa^P} \rangle_g := \sum_{\beta \in H_2^+(V,\mathbb{Z})} \langle \boldsymbol{\sigma^r \tau^m \kappa^P} \rangle_{g,\beta} \; q^\beta,$$

where q is a formal variable.

In the sequel we will consider the following collection of formal variables: $\mathbf{x} = (x^\nu)$, $\mathbf{t} = (t_i^\mu)$, $\mathbf{s} = (s_a^\alpha)$, $i > 0$, $a \geq 0$, $\nu, \mu, \alpha \in A$. These variables have the following degrees: $|x^\nu| = |\nu| - 2$, $|t_i^\mu| = 2(i-1) + |\mu|$, and $|s_a^\alpha| = 2a + |\alpha|$. Note that the $\mathbb{Z}/2\mathbb{Z}$-degree is determined by the upper index. Let

$$\mathbf{x^r} := \prod_\nu (x^\nu)^{r_\nu}, \quad \mathbf{t^m} := \prod_{d,\mu} (t_d^\mu)^{m_{d,\mu}}, \quad \mathbf{s^P} := \prod_{a,\alpha} (s_a^\alpha)^{p_{a,\alpha}}.$$

Again one has to exercise care in case there are variables of odd degree. In this case we order the products above so that

x^{ν_1} preceeds x^{ν_2} if $\nu_1 \geq \nu_2$;
$t_{d_1}^{\mu_1}$ preceeds $t_{d_2}^{\mu_2}$ if $d_1 > d_2$, or $d_1 = d_2$ and $\mu_1 \geq \mu_2$;
$s_{a_1}^{\alpha_1}$ preceeds $s_{a_2}^{\alpha_2}$ if $a_1 > a_2$, or $a_1 = a_2$ and $\alpha_1 \geq \alpha_2$.
That is, we require the order on the products to be the opposite to the order on the intersection numbers.

Definition 5.3. We define $\mathcal{K}_g \in \mathbb{C}[[\Lambda, \mathbf{x}, \mathbf{t}, \mathbf{s}]]$ by

$$\mathcal{K}_g(\mathbf{x}, \mathbf{t}, \mathbf{s}) := \sum_{\mathbf{r,m,p}} \langle \boldsymbol{\sigma^r \tau^m \kappa^P} \rangle_g \frac{\mathbf{s^P}}{\mathbf{p}!} \frac{\mathbf{t^m}}{\mathbf{m}!} \frac{\mathbf{x^r}}{\mathbf{r}!}$$

where $\mathbf{p} := \prod_\nu p_\nu$ and $\mathbf{p}! := \prod_\nu p_\nu!$ (and similarly for $\mathbf{m}$ and $\mathbf{r}$.

Remark . In the above definition one could have chosen an arbitrary ordering for the intersection numbers, and then chosen the opposite ordering on the corresponding variables.

The various degrees chosen for the variables together with the dimensions of the cohomology classes and the virtual fundamental class insures the $\mathcal{K}_g$ has degree $2(3-d)(1-g)$.

Note that $\mathcal{K}(\mathbf{x}, \mathbf{t}, \mathbf{0}) = \mathcal{F}(\mathbf{x}, \mathbf{t})$, the standard large phase space potential if one sets $x^\nu = t_0^\nu$. Similarly, $\mathcal{K}(\mathbf{x}, \mathbf{0}, \mathbf{s}) = \mathcal{G}(\mathbf{x}, \mathbf{s})$, the potential of the family of CohFTs determined by the κ classes including the terms with $2g - 2 + n \leq 0$.

Theorem 5.4. *Let* $\mathbf{t}(\mathbf{s})$ *be determined by the following equation in* $H^\bullet(V)$:

$$(2) \qquad e_0 - \sum_{d \geq 1} \theta^{d-1} t_d^\mu e_\mu = \exp\left(-\sum_{a \geq 0} \theta^a s_a^\alpha e_\alpha\right),$$

where θ *is an even formal parameter. Then* $\mathcal{F}_g(\mathbf{x}, \mathbf{t}(\mathbf{s})) = \mathcal{G}_g(\mathbf{x}, \mathbf{s})$ *for every* $g \geq 0$.

Remark . In case when $V = pt$ Thm. 5.4 reduces to Thm. 4.1 from [40]. The polynomials $t_a(\mathbf{s})$ are the Schur polynomials.

We will prove the above theorem in a sequence of lemmata.

Lemma 5.5. *Let* I *and* J *be two sets such that* $I \cap J = \{1, \ldots, n\}$ *and* $I \cup J = \{1, \ldots, n + N\}$. *Let* $I' := I - \{1, \ldots, n\}$ *and* $J' := J - \{1, \ldots, n\}$. *Consider the following commutative diagram*

$$\begin{array}{ccc}
\overline{\mathcal{M}}_{g,I}(V, \beta) & \xleftarrow{\quad \rho \quad} & \overline{\mathcal{M}}_{g,n+N}(V, \beta) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\overline{\mathcal{M}}_{g,n}(V, \beta) & \xleftarrow{\quad \rho \quad} & \overline{\mathcal{M}}_{g,J}(V, \beta),
\end{array}$$

where the horizontal morphisms ρ *"forget" the marked points from* J', *and the vertical morphisms* π *"forget" the marked points from* I'. *The morphisms* π *and* ρ *are flat, and*

$$\rho^* \pi_* = \pi_* \rho^* \quad and \quad \rho_* \pi^* = \pi^* \rho_*$$

Proof. The morphisms π and ρ are flat as compositions of flat morphisms. Let I be $\{1, \ldots, n+1\}$, and J be $\{1, \ldots, n, n+2\}$. Then the commutative diagram above is close to a fibered square in the sense of Def. 1.1 (cf. [1]). Therefore one has $\rho^* \pi_* = \pi_* \rho^*$ and $\rho_* \pi^* = \pi^* \rho_*$. Iterating, one obtains the statement of the lemma. $\square$

Consider the universal curve $\pi : \overline{\mathcal{M}}_{g,n+1}(V, \beta) \to \overline{\mathcal{M}}_{g,n}(V, \beta)$. It has n canonical sections σ_i, and each of these sections is a regular embedding of codimension one. Therefore, the image of each of these sections determines a Cartier divisor on $\overline{\mathcal{M}}_{g,n+1}(V, \beta)$. We denote the corresponding Chern classes by $D_{i,n+1} \in H^2(\overline{\mathcal{M}}_{g,n+1}(V, \beta))$. Equivalently, $D_{i,n+1} = \sigma_{i*}1$. The equalities below hold in $H^\bullet(\overline{\mathcal{M}}_{g,n+1}(V, \beta))$:

$$D_{i,n+1} D_{j,n+1} = 0 \quad \text{if } i \neq j,$$
$$\psi_i D_{i,n+1} = \psi_{n+1} D_{i,n+1} = 0.$$

In addition, $\sigma_i^* D_{i,n+1} = -\psi_i$.

Let $\pi : \overline{\mathcal{M}}_{g,n+1}(V, \beta) \to \overline{\mathcal{M}}_{g,n}(V, \beta)$ be the universal curve. In the next two lemmas we will use the following properties. Firstly, $\pi^* \psi_i =$

$\psi_i - D_{i,n+1}$ proved in [18, Prop. 11]. It follows that $\pi^*\psi_i^a = \psi_i^a + (-1)^a D_{i,n+1}^a$. Secondly, $\pi^* ev_i^* = ev_i^*$.

The following lemma and its proof are similar to those in [1, Sec. 1].

Lemma 5.6. *If $\gamma \in H^\bullet(V)$, then $\pi^*\kappa_a(\gamma) = \kappa_a(\gamma) - \psi_{n+1}^a ev_{n+1}^*\gamma$.*

Proof. Consider the following commutative diagram close to a fibered square:

$$
\begin{array}{ccc}
\overline{\mathcal{M}}_{g,n+1}(V,\beta) & \xleftarrow{\ \ \rho\ \ } & \overline{\mathcal{M}}_{g,n+2}(V,\beta) \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
\overline{\mathcal{M}}_{g,n}(V,\beta) & \xleftarrow{\ \ \rho\ \ } & \overline{\mathcal{M}}_{g,J}(V,\beta),
\end{array}
$$

where $J = \{1, \ldots, n, n+2\}$. Let $\sigma : \overline{\mathcal{M}}_{g,n+1}(V,\beta) \to \overline{\mathcal{M}}_{g,n+2}(V,\beta)$ associated to the $(n+1)^{\text{st}}$ marked point. One has

$$
\begin{aligned}
\pi^*\kappa_a(\gamma) &= \pi^*\rho_*(\psi_{n+2}^{a+1} ev_{n+2}^*\gamma) = \rho_*\pi^*(\psi_{n+2}^{a+1} ev_{n+2}^*\gamma) \\
&= \rho_*(\psi_{n+2}^{a+1} ev_{n+2}^*\gamma) + (-1)^{a+1}\rho_*(D_{n+1,n+2}^{a+1}, ev_{n+2}^*\gamma) \\
&= \kappa_a(\gamma) + (-1)^{a+1}\rho_*\sigma_*\sigma^*(D_{n+1,n+2}^a, ev_{n+2}^*\gamma) = \kappa_a(\gamma) - \psi_{n+1}^a ev_{n+1}^*\gamma.
\end{aligned}
$$
$\square$

Definition 5.7. Let $\gamma \in H^\bullet(V)$. Define the homomorphism $\varkappa_a(\gamma) : H^\bullet(\overline{\mathcal{M}}_{g,n+1}(V,\beta)) \to H^\bullet(\overline{\mathcal{M}}_{g,n}(V,\beta))$ by

$$
\varkappa_a(\gamma)(x) := \pi_*(\psi_{n+1}^{a+1} ev_{n+1}^*\gamma\ x).
$$

Note that $\varkappa_a(\gamma)(1) = \kappa_a(\gamma)$.

Lemma 5.8. *Assume that all $a_i > 0$ for $i = 1, \ldots, N$. Then*

$$
\varkappa_{a_1-1}(\gamma_1) \ldots \varkappa_{a_N-1}(\gamma_N)(x) = \pi_{N*}(\psi_{n+1}^{a_1} ev_{n+1}^*\gamma_1 \ldots \psi_{n+N}^{a_N} ev_{n+N}^*\gamma_N\ x),
$$

where $\pi_N : \overline{\mathcal{M}}_{g,n+N}(V,\beta) \to \overline{\mathcal{M}}_{g,n}(V,\beta)$ "forgets" the last N marked points.

Proof. We proceed by induction. When $N = 1$ the statement of the lemma is trivial. Assume that the statement is true for N and prove it for $N+1$. We denote by π' the universal curve $\overline{\mathcal{M}}_{g,n+N+1}(V,\beta) \to \overline{\mathcal{M}}_{g,n+N}(V,\beta)$. One has

$$
\begin{aligned}
&\varkappa_{a_1-1}(\gamma_1) \ldots \varkappa_{a_{N+1}-1}(\gamma_{N+1})(x) \\
&= \varkappa_{a_1-1}(\gamma_1) \ldots \varkappa_{a_N-1}(\gamma_N)(\pi'_*(\psi_{n+N+1}^{a_{N+1}} ev_{n+N+1}^*\gamma_{N+1}\ x)) \\
&= \pi_{N*}(\psi_{n+1}^{a_1} ev_{n+1}^*\gamma_1 \ldots \psi_{n+N}^{a_N} ev_{n+N}^*\gamma_N\ \pi'_*(\psi_{n+N+1}^{a_{N+1}} ev_{n+N+1}^*\gamma_{N+1}\ x)) \\
&= \pi_{N+1*}((\psi_{n+1} - D_{n+1,n+N+1})^{a_1} ev_{n+1}^*\gamma_1 \ldots \\
&\qquad (\psi_{n+N} - D_{n+N,n+N+1})^{a_N} ev_{n+N}^*\gamma_N\ \psi_{n+N+1}^{a_{N+1}} ev_{n+N+1}^*\gamma_{N+1}\ x)) \\
&= \pi_{N+1*}(\psi_{n+1}^{a_1} ev_{n+1}^*\gamma_1 \ldots \psi_{n+N}^{a_N} ev_{n+N}^*\gamma_N\ \psi_{n+N+1}^{a_{N+1}} ev_{n+N+1}^*\gamma_{N+1}\ x).
\end{aligned}
$$
$\square$

It follows from Lem. 5.8 that the operators $\varkappa_a(\gamma)$ super-commute.

Remark . If some if the numbers a_i are equal to zero, then Lem. 5.8 does not necessarily hold.

Lemma 5.9. *Assume the conditions of Lem. 5.8. Then*

$$\pi_{N*}(\psi_{n+1}^{a_1}ev_{n+1}^*\gamma_1 \dots \psi_{n+N}^{a_N}ev_{n+N}^*\gamma_N)\kappa_a(\gamma)$$
$$=\pi_{N+1*}(\psi_{n+1}^{a_1}ev_{n+1}^*\gamma_1 \dots \psi_{n+N}^{a_N}ev_{n+N}^*\gamma_N \ \psi_{n+1+N}^{a+1}ev_{n+N+1}^*\gamma)$$
$$-\sum_{i=1}^{k}(-1)^{|\gamma_{i+1}\dots\gamma_N||\gamma|}\pi_{N*}(\psi_{n+1}^{a_1}ev_{n+1}^*\gamma_1 \dots \psi_{n+i}^{a_i+a}ev_{n+i}^*(\gamma_i\gamma) \dots \psi_{n+N}^{a_N}ev_{n+N}^*\gamma_N).$$

Proof. We proceed by induction. Let $N = 1$, and $\pi_N = \pi$. Then, using Lem. 5.6 and Lem. 5.8, one gets

$$\pi_*(\psi_{n+1}^{a_1}ev_{n+1}^*\gamma_1\pi^*\kappa_a(\gamma)) = \varkappa_{a_1-1}(\gamma_1)(\kappa_a(\gamma)) - \pi_*(\psi_{n+1}^{a_1+a}ev_{n+1}^*(\gamma_1\gamma)).$$

This proves the statement of the lemma when $N = 1$. Now assume that the statement is true for $N' = N - 1$ and prove it for N. Denote by π'_N the natural morphism $\overline{\mathcal{M}}_{g,n+N}(V,\beta) \to \overline{\mathcal{M}}_{g,n+1}(V,\beta)$.

$$\pi_{N*}(\psi_{n+1}^{a_1}ev_{n+1}^*\gamma_1 \dots \psi_{n+N}^{a_N}ev_{n+N}^*\gamma_N)\kappa_a(\gamma)$$
$$=\pi_*(\psi_{n+1}^{a_1}ev_{n+1}^*\gamma_1\pi'_{N*}(\psi_{n+2}^{a_2}ev_{n+2}^*\gamma_2 \dots \psi_{n+N}^{a_N}ev_{n+N}^*\gamma_N)$$
$$(\kappa_a(\gamma) - \psi_{n+1}^{a}ev_{n+1}^*\gamma)).$$

The rest follows applying the induction hypothesis to the product

$$\pi'_{N*}(\psi_{n+2}^{a_2}ev_{n+2}^*\gamma_2 \dots \psi_{n+N}^{a_N}ev_{n+N}^*\gamma_N)\kappa_a(\gamma),$$

and using Lem. 5.8. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 5.9 provides a recursion relation for the intersection numbers of the ψ and the κ classes. Let $\{e_\alpha\}$, $\alpha = 0, \dots, r$, be the chosen basis of $H^\bullet(V)$. Define $c_{\alpha_1,\dots,\alpha_j}^\mu$ by the formula

$$e_{\alpha_1} \dots e_{\alpha_j} = c_{\alpha_1,\dots,\alpha_j}^\mu e_\mu.$$

(We assume summation over the repeating indices.) In particular, $c_\alpha^\mu = \delta_\alpha^\mu$.

The following recursion relation follows from Lem. 5.9.

$$\langle\boldsymbol{\sigma^r}\tau_{d_1,\mu_1} \dots \tau_{d_k,\mu_k}\kappa_{a,\alpha}\boldsymbol{\kappa^P}\rangle_{g,\beta} = \langle\boldsymbol{\sigma^r}\tau_{d_1,\mu_1} \dots \tau_{d_k,\mu_k}\tau_{a+1,\alpha}\boldsymbol{\kappa^P}\rangle_{g,\beta}$$
$$-\sum_{i=1}^{k}(-1)^{|e_{\mu_{i+1}}\dots e_{\mu_k}||e_\alpha|}c_{\mu_i,\alpha}^\mu\langle\boldsymbol{\sigma^r}\tau_{d_1,\mu_1} \dots \tau_{d_i+a,\mu} \dots \tau_{d_k,\mu_k}\boldsymbol{\kappa^P}\rangle_{g,\beta}.$$

Note that the above relations also holds if one replaces $\langle\dots\rangle_{g,\beta}$ with $\langle\dots\rangle_g$.

It turns out that the equation above implies that for each $g \geq 0$

$$(3) \qquad \frac{\partial \mathcal{K}_g}{\partial s_a^\alpha} = \frac{\partial \mathcal{K}_g}{\partial t_{a+1}^\alpha} - \sum_{i=1}^\infty c_{\nu,\alpha}^\mu t_i^\nu \frac{\partial \mathcal{K}_g}{\partial t_{i+a}^\mu}.$$

We leave to the reader to check that all signs agree.

Let us introduce the following standard notation:

$$\tilde{t}_i^\nu := \begin{cases} t_i^\nu & \text{unless } i = 1 \text{ and } \nu = 0, \\ t_1^0 - 1 & \text{if } i = 1 \text{ and } \nu = 0. \end{cases}$$

Then one can rewrite (3) as

$$(4) \qquad \frac{\partial \mathcal{K}_g}{\partial s_a^\alpha} = -\sum_{i=1}^\infty c_{\nu,\alpha}^\mu \tilde{t}_i^\nu \frac{\partial \mathcal{K}_g}{\partial t_{i+a}^\mu}.$$

Proof. (of Thm. 5.4.) We assume that $\mathbf{t}(\mathbf{s})$ is determined by (2). It follows that $\mathbf{t}(\mathbf{0}) = \mathbf{0}$, and

$$-\sum_{d \geq 1} \theta^{d-1} \frac{\partial t_d^\mu}{\partial s_a^\alpha} e_\mu = -\theta^a e_\alpha \exp\left(-\sum_{a_1 \geq 0} \theta^{a_1} s_{a_1}^{\alpha_1} e_{\alpha_1}\right) = \sum_{d \geq a+1} \theta^{d-1} \tilde{t}_{d-a}^\mu e_\mu e_\alpha.$$

It follows that for each d, a, and α such that $d \geq a+1$ one has

$$(5) \qquad \frac{\partial t_d^\mu}{\partial s_a^\alpha} e_\mu = -\tilde{t}_{d-a}^\nu e_\nu e_\alpha \quad \text{or, equivalently,} \quad \frac{\partial t_d^\mu}{\partial s_a^\alpha} = -\tilde{t}_{d-a}^\nu c_{\nu,\alpha}^\mu,$$

and $\partial t_d^\mu / \partial s_a^\alpha = 0$ if $d \leq a$.

Consider the function $\mathcal{K}_g(\mathbf{x}, \mathbf{t}(\mathbf{s}_0 + \mathbf{s}), -\mathbf{s})$, where $\mathbf{s}_0$ is a constant. Differentiating it with respect to s_a^α provides using (5) and (4)

$$\frac{\partial}{\partial s_a^\alpha} [\mathcal{K}_g(\mathbf{x}, \mathbf{t}(\mathbf{s}_0 + \mathbf{s}), -\mathbf{s})]$$

$$= -\frac{\partial \mathcal{K}_g}{\partial s_a^\alpha}(\mathbf{x}, \mathbf{t}(\mathbf{s}_0 + \mathbf{s}), -\mathbf{s}) + \sum_{d \geq 1} \frac{\partial t_d^\mu}{\partial s_a^\alpha}(\mathbf{s}_0 + \mathbf{s}) \frac{\partial \mathcal{K}_g}{\partial t_d^\mu}(\mathbf{x}, \mathbf{t}(\mathbf{s}_0 + \mathbf{s}), -\mathbf{s})$$

$$= -\frac{\partial \mathcal{K}_g}{\partial s_a^\alpha}(\mathbf{x}, \mathbf{t}(\mathbf{s}_0 + \mathbf{s}), -\mathbf{s}) - \sum_{d \geq 1} c_{\nu,\alpha}^\mu \tilde{t}_d^\nu(\mathbf{s}_0 + \mathbf{s}) \frac{\partial \mathcal{K}_g}{\partial t_{d+a}^\mu}(\mathbf{x}, \mathbf{t}(\mathbf{s}_0 + \mathbf{s}), -\mathbf{s}) = 0.$$

Therefore $\mathcal{K}_g(\mathbf{x}, \mathbf{t}(\mathbf{s}_0 + \mathbf{s}), -\mathbf{s})$ does not depend on $\mathbf{s}$. It follows that for all values $\mathbf{s}_1$ and $\mathbf{s}_2$ one has

$$\mathcal{K}_g(\mathbf{x}, \mathbf{t}(\mathbf{s}_1 + \mathbf{s}_2), \mathbf{0}) = \mathcal{K}_g(\mathbf{x}, \mathbf{t}(\mathbf{s}_1), \mathbf{s}_2) = \mathcal{K}_g(\mathbf{x}, \mathbf{0}, \mathbf{s}_1 + \mathbf{s}_2).$$

In particular, $\mathcal{F}_g(\mathbf{x}, \mathbf{t}(\mathbf{s})) = \mathcal{G}_g(\mathbf{x}, \mathbf{s})$ for every $g \geq 0$. $\qquad\square$

Remark . Note that the condition $\mathbf{t}(\mathbf{s}) = \mathbf{0}$ and (5) are equivalent to (2), and determine $\mathbf{t}(\mathbf{s})$ completely. Note also that the coordinate change given by (2) is invertible.

Remark . The function $\mathbf{t}(\mathbf{s})$ has the following Taylor coefficients:

$$\frac{\partial^k t_d^\mu}{\partial s_{a_1}^{\alpha_1} \ldots \partial s_{a_k}^{\alpha_k}}\mid_{\mathbf{s}=\mathbf{0}} = (-1)^{k+1} c_{\alpha_1,\ldots,\alpha_k}^\mu \delta_{d,a_1+\ldots+a_k+1}.$$

6. Topological Recursion Relations

In this section we will derive the topological recursion relations for $\mathcal{G}_0$ and $\mathcal{G}_1$ using the change of coordinates formula (2). In [23] we represented the cohomology classes by graphs to obtain the topological recursion relations when V is a convex variety and genus $g = 0$. However, we were not able to extend this technique to the general case since it is not clear that one can pull back homology classes w.r.t. $\mu(\Gamma)$ from Sec. 2.

We will use the fact that $\mathcal{G}_g(\mathbf{x}, \mathbf{s}) = \mathcal{F}_g(\mathbf{x}, \mathbf{t}(\mathbf{s}))$, where $\mathbf{t}(\mathbf{s})$ is determined by (2). Notice that $\partial \mathcal{G}_g / \partial x^\alpha = \partial \mathcal{F}_g / \partial x^\alpha$ since the coordinate change $\mathbf{t}(\mathbf{s})$ does not depend on $\mathbf{x}$. We will raise and lower indices in the usual manner.

Proposition 6.1. *Let* $a \geq 1$. *Then*

$$\frac{\partial^3 \mathcal{G}_0}{\partial s_a^\alpha \partial x^\mu \partial x^\nu} = \frac{\partial^2 \mathcal{G}_0}{\partial s_{a-1}^\alpha \partial x^\rho} \frac{\partial^3 \mathcal{G}_0}{\partial x_\rho \partial x^\mu \partial x^\nu}.$$

Proof. Applying the chain rule one gets:

$$\frac{\partial^3 \mathcal{G}_0}{\partial s_a^\alpha \partial x^\mu \partial x^\nu} = \sum_{d \geq a+1} \frac{\partial t_d^\xi}{\partial s_a^\alpha} \frac{\partial^3 \mathcal{F}_0}{\partial t_d^\xi \partial x^\mu \partial x^\nu} = \sum_{d \geq a+1} \frac{\partial t_d^\xi}{\partial s_a^\alpha} \frac{\partial^2 \mathcal{F}_0}{\partial t_{d-1}^\xi \partial x^\rho} \frac{\partial^3 \mathcal{F}_0}{\partial x_\rho \partial x^\mu \partial x^\nu}.$$

The second equation uses that $\mathcal{F}_0$ satisfies the topological recursion relations. Similarly,

$$\frac{\partial^2 \mathcal{G}_0}{\partial s_{a-1}^\alpha \partial x^\rho} \frac{\partial^3 \mathcal{G}_0}{\partial x_\rho \partial x^\mu \partial x^\nu} = \sum_{d \geq a} \frac{\partial t_d^\xi}{\partial s_{a-1}^\alpha} \frac{\partial^2 \mathcal{F}_0}{\partial t_d^\xi \partial x^\rho} \frac{\partial^3 \mathcal{F}_0}{\partial x_\rho \partial x^\mu \partial x^\nu}.$$

Equation (5) implies that $\partial t_d^\xi / \partial s_{a-1}^\alpha = \partial t_{d+1}^\xi / \partial s_a^\alpha$, and the proposition follows. $\square$

Proposition 6.2. *Let* $|\alpha| \leq 2$. *Then*

(6) $$\frac{\partial^3 \mathcal{G}_0}{\partial s_0^\alpha \partial x^\mu \partial x^\nu} = \mathbf{D}_\alpha(\frac{\partial \mathcal{G}_0}{\partial x^\rho}) \frac{\partial^3 \mathcal{G}_0}{\partial x_\rho \partial x^\mu \partial x^\nu} + c_{\rho,\alpha}^\xi x_\xi \frac{\partial^3 \mathcal{G}_0}{\partial x_\rho \partial x^\mu \partial x^\nu}$$

where the differential operator $\mathbf{D}_\alpha$ *is the* $\mathbb{C}[[\mathbf{x}, \mathbf{t}, \mathbf{s}]]$*-linear operator defined by*

$$(7) \qquad \mathbf{D}_\alpha \, q^\beta := q^\beta \int_\beta e_\alpha$$

for all α.

Proof. We use the chain rule, (5), and the topological recursion relations for $\mathcal{F}_0$:

$$\frac{\partial^3 \mathcal{G}_0}{\partial s_0^\alpha \partial x^\mu \partial x^\nu} = -\sum_{d \geq 1} c_{\xi,\alpha}^\zeta \tilde{t}_d^\xi \frac{\partial^2 \mathcal{F}_0}{\partial t_{d-1}^\zeta \partial x^\rho} \frac{\partial^3 \mathcal{F}_0}{\partial x_\rho \partial x^\mu \partial x^\nu}$$

$$= \mathbf{D}_\alpha \left(\frac{\partial \mathcal{F}_0}{\partial x^\rho} \right) \frac{\partial^3 \mathcal{F}_0}{\partial x_\rho \partial x^\mu \partial x^\nu} + c_{\rho,\alpha}^\xi x_\xi \frac{\partial^3 \mathcal{F}_0}{\partial x_\rho \partial x^\mu \partial x^\nu}.$$

In the second equation we used the divisor equation for $\mathcal{F}$ [19, 2.6]. $\quad\square$

Remark . The first term of the right hand side in (6) contributes only when $|e_\alpha| = 2$. When $\alpha = 0$ one can get (6) from

$$\frac{\partial \mathcal{G}_0}{\partial s_0^0} = \sum_\rho x^\rho \frac{\partial \mathcal{G}_0}{\partial x^\rho} - 2\mathcal{G}_0.$$

This equation can be derived using the dilaton equation for $\mathcal{F}_0$ [19, 2.7].

Similarly, one can derive the topological recursion relations for $\mathcal{G}$ in genus 1 using known topological recursion relations for $\mathcal{F}$. We state the results without proofs.

Proposition 6.3. *Let* $a \geq 1$. *Then*

$$\frac{\partial \mathcal{G}_1}{\partial s_a^\alpha} = \frac{\partial^2 \mathcal{G}_0}{\partial s_{a-1}^\alpha \partial x^\rho} \frac{\partial \mathcal{G}_1}{\partial x_\rho} + \frac{1}{24} \frac{\partial^3 \mathcal{G}_0}{\partial s_{a-1}^\alpha \partial x^\rho \partial x_\rho}.$$

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